Answer model exam Lie groups in Physics of January 31, 2020

1

1a) Lie algebra is vector space L endowed with a product $[a,b]$, s.t. $[a,b] \in L \ \forall a,b \in L$, Lie product is linear, antisymmetric and satisfies Jacobi id.

1b) invariant subalgebra: $\forall a \in L, \forall b \in L' \subset L: [a,b] \in L'$

simple Lie algebra: does not possess a proper invariant subalgebra. & not Abelian.

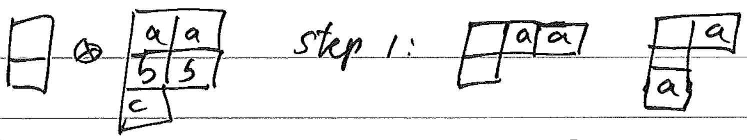
1c) examples of Lie gp. that are not simple: $O(3)$, Poincaré gp.

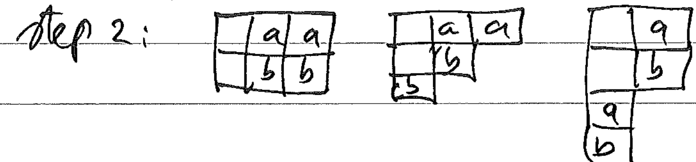
1d) If coset space G/H forms a gp. then H is an invariant subgp. Then algebra of H forms an invariant subalg of G.

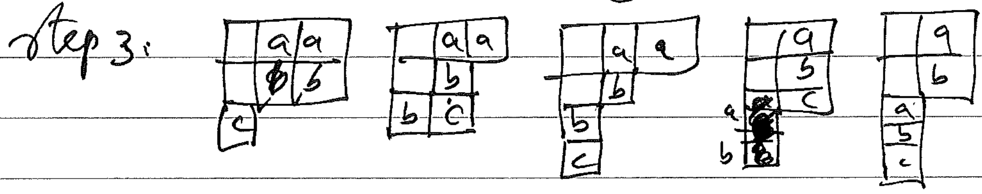
2

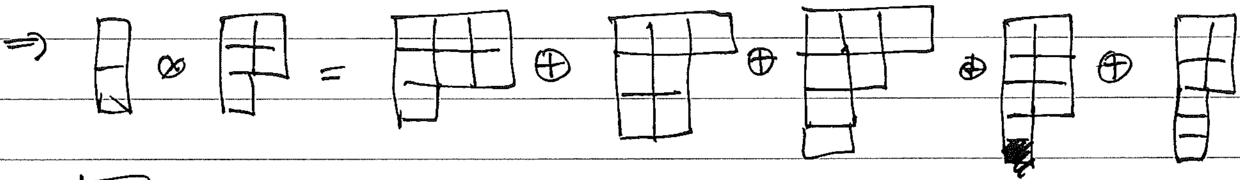
2a) Allowed words: aabbc, aabcb, ababc, abacb, abcab

2b)

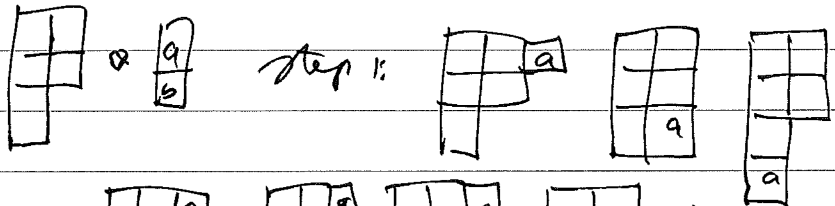


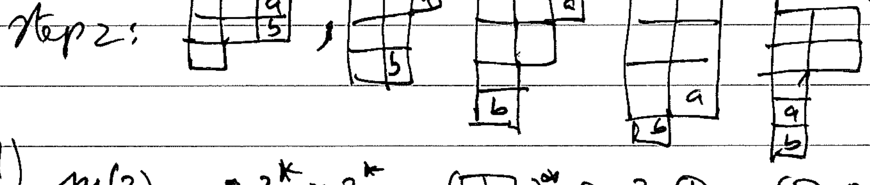






2c)





same as 2b

2d) $su(3): \ 3^* \times 3^* = (\square\square)^* \oplus 3 \oplus 0 \oplus 0 \oplus 0 = 6^* \oplus 3$

$su(4): \ 20^* \times 6 = (\square\square\square)^+ \oplus (\square)^+ \oplus \square \oplus 4^* \oplus 0$

$= 60^* \oplus (36)^* \oplus 20 \oplus 4^*$

Answer model exam Lie groups in Physics of January 31, 2020

3 3a) $O(3) \cong \mathbb{Z}_2 \otimes SO(3)$? All elements of $O(3)$ have det ± 1

" $SO(3)$ have det $+1$.

Define $P = \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix}$ det$P = -1$, so all elements of $O(3)$ are either a rotation or $P \times$ rotation.

Then $O(3) = \{ R, PR \mid \forall R \in SO(3) \}$.

The mapping $R \mapsto (+1, R)$ $PR \mapsto (-1, R)$

provides a 1-1 mapping from $O(3)$ to $\mathbb{Z}_2 \otimes SO(3)$

3b) $x_i \to x_i' = D^V(g)_{ij} x_j \Rightarrow x_k' y_l' = D^V_{ki} D^V_{lj} x_i y_j$

Define $T_{kl} \equiv D^V_{ki} D^V_{lj} (x_i y_j \pm x_j y_i)$

Consider $T_{lk} = D^V_{li} D^V_{kj} (x_i y_j \pm x_j y_i)$ } change order of D^V's

$= D^V_{kj} D^V_{li} (x_i y_j \pm x_j y_i)$ } rename $i \Leftrightarrow j$

$= D^V_{ki} D^V_{lj} (x_j y_i \pm x_i y_j)$

$= \pm D^V_{ki} D^V_{lj} (x_i y_j \pm x_j y_i) = \pm T_{kl}$.

Hence (anti)symmetric remains (anti)symmetric after $D^V \in SO(3)$

3c) Defining rep of $O(3)$ is irreducible since any direction in $\mathbb{R}^3$ can be obtained from any other direction in $\mathbb{R}^3$ by an $O(3)$ trafo. When restricting to $O(2)$ there is a fixed direction that doesn't transform $\Rightarrow$ 1D invariant subspace, hence not irred.

4 4a) $(M^{\mu\nu})^\alpha{}_\beta = i(g^{\nu\alpha} g^\nu{}_\beta - g^{\nu\alpha} g^\mu{}_\beta)$

$-(M^{20})^\alpha{}_\beta = -i(g^{2\alpha} g^0{}_\beta - g^{0\alpha} g^2{}_\beta) = \begin{pmatrix} 0 & 0 & +i & 0 \\ 0 & 0 & 0 & 0 \\ +i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$(M^{02})^\alpha{}_\beta = -(M^{20})^\alpha{}_\beta$. $g^{22} = -1$ $g^0{}_0 = 1$ $g^2{}_2 = 1$

$(M^{12})^\alpha{}_\beta = i(g^{1\alpha} g^2{}_\beta - g^{2\alpha} g^1{}_\beta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$g^{11} = -1$ $g^2{}_2 = 1$ $g^1{}_1 = 1$

Answer model exam Lie groups in physics of January 31, 2020.

4b) $(M^{20})^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, but $(M^{20})^0 = \mathbb{1}$

$$\exp(-i\chi M^{02}) = \exp(i\chi M^{20}) = \sum_{n=0}^{\infty} \frac{1}{n!} (i\chi M^{20})^n$$

even terms minus $n=0$ term: $\sum_{n=1}^{\infty} \frac{1}{2n!} (-1)^n \chi^{2n} (M^{20})^{2n}$

$$= \sum_{n=1}^{\infty} \frac{1}{2n!} \cancel{(-1)^n} \chi^{2n} \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} = \begin{pmatrix} \cosh\chi - 1 & & & \\ & 0 & & \\ & & \cosh\chi - 1 & \\ & & & 0 \end{pmatrix}$$

odd terms: $\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (-1)^n \chi^{2n+1} (M^{20})^{2n} \cdot iM^{20}$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \chi^{2n+1} \cdot \underbrace{\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} = \begin{pmatrix} 0 & 0 & \sinh\chi & 0 \\ 0 & 0 & 0 & 0 \\ \sinh\chi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\exp(-i\chi M^{02}) = \begin{pmatrix} \cosh\chi & 0 & \sinh\chi & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\chi & 0 & \cosh\chi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{boost in } y \text{ direction}$$

weighting

	1a	6	2a	7	3a	7	4a	7
	1b	6	2b	8	3b	8	4b	8
	1c	6	2c	7	3c	7		
	1d	6	2d	7				

Result $= \frac{\sum \text{points}}{10} + 1$